J. Fluid Mech. (2004), *vol.* 499, *pp.* 231–255.
DOI: 10.1017/S0022112003007055 Printed in the United Kingdom

Turbulent diffusion in rapidly rotating flows with and without stable stratification

By C. CAMBON[1], F. S. GODEFERD[1], F. C. G. A. NICOLLEAU[2] AND J. C. VASSILICOS[3]

[1]Laboratoire de Mécanique des Fluides et d'Acoustique UMR 5509, École Centrale de Lyon, 69134 Ecully Cedex, France

[2]Department of Mechanical Engineering, University of Sheffield, Sheffield S1 3JD, UK

[3]Department of Aeronautics, Imperial College, Prince Consort Road, London SW7 2BY, UK

(Received 29 May 2002 and in revised form 12 September 2003)

In this work, three different approaches are used for evaluating some Lagrangian properties of homogeneous turbulence containing anisotropy due to the application of a stable stratification and a solid-body rotation. The two external frequencies are the magnitude of the system vorticity 2Ω, chosen vertical here, and the Brunt–Väisälä frequency N, which gives the strength of the vertical stratification. Analytical results are derived using linear theory for the Eulerian velocity correlations (single-point, two-time) in the vertical and the horizontal directions, and Lagrangian ones are assumed to be equivalent, in agreement with an additional Corrsin assumption used by Kaneda (2000). They are compared with results from the kinematic simulation model (KS) by Nicolleau & Vassilicos (2000), which also incorporates the wave–vortex dynamics inherited from linear theory, and directly yields Lagrangian correlations as well as Eulerian ones. Finally, results from direct numerical simulations (DNS) are obtained and compared for the rotation-dominant case $B = 2\Omega/N = 10$, the stratification-dominant case $B = 1/10$, the non-dispersive case $B = 1$, and pure stratification $B = 0$ and pure rotation $N = 0$. The last situation is shown to be singular with respect to the mixed stratified/rotating ones. We address the question of the validity of Corrsin's simplified hypothesis, which states the equivalence between Eulerian and Lagrangian correlations. Vertical correlations are found to follow this postulate, but not the horizontal ones. Consequences for the vertical and horizontal one-particle dispersion are examined. In the analytical model, the squared excursion lengths are calculated by time integrating the Lagrangian (equal to the Eulerian) two-time correlations, according to Taylor's procedure. These quantities are directly computed from fluctuating trajectories by both KS and DNS. In the case of pure rotation, the analytical procedure allows us to relate Brownian t-asymptotic laws of dispersion in both the horizontal and vertical directions to the angular phase-mixing properties of the inertial waves. If stratification is present, the inertia–gravity wave dynamics, which affects the vertical motion, yields a suppressed vertical diffusivity, but not a suppressed horizontal diffusivity, since part of the horizontal velocity field escapes wavy motion.

1. Introduction

Rapid rotation and stable stratification strongly affect turbulent diffusion. Instances of rapidly rotating turbulence with or without stable stratification may be found in many geophysical, industrial and laboratory flows. Direct numerical simulations (DNS) and laboratory experiments have revealed that rapid rotation and stable stratification have highly anisotropic effects on turbulent diffusion (e.g. Britter *et al.* 1983; Vincent, Michaud & Meneguzzi 1996; Kimura & Herring 1996, 1999; Kaneda & Ishida 2000): turbulent diffusion can be suppressed in one direction but not in others. Surprisingly, in spite of observations in nature, it is difficult to find results of laboratory experiments on turbulent diffusion for rotating flows both with and without stable stratification. (The only experimental work quoted in the recent review of Lagrangian aspects by Yeung (2002) is Jacquin *et al.* (1990), but turbulent diffusion is only marginally discussed, and not measured, in this paper about rotating turbulence.)

Recent modelling and theoretical approaches by Kaneda & Ishida (2000), Nicolleau & Vassilicos (2000), Kaneda (2000), and Nicolleau & Vassilicos (2003) have been able to explain and predict the suppression of one-particle† turbulent diffusion along the direction of stratification (vertical) in stably stratified turbulence with and without rapid rotation. All these approaches have in common the use of solutions of the linearized equations of motion. However, Kaneda & Ishida (2000) and Kaneda (2000) base their predictions on Corrsin's conjecture whereas the kinematic simulations (KS) of Nicolleau & Vassilicos (2000) do not. Corrsin's conjecture allows the estimation of two-time Lagrangian velocity correlations from a calculation of two-time Eulerian velocity correlations.

Two-time Lagrangian velocity correlations are central quantities in turbulent diffusion because of Taylor's relation (equation (2.2) in the following section), which expresses the mean-square displacements of fluid particles as a double integral over time of two-time Lagrangian velocity correlations. Taylor's relation has important physical consequences for turbulent diffusion. In isotropic turbulence, the mean-square displacement behaves proportionally to t^2 for short times, in agreement with a *ballistic* regime, and evolves proportionaly to t for larger times, in agreement with a *Brownian* regime (Taylor 1921). Taylor's relation can also imply depletion of turbulence diffusion, caused, for example, by vortex trapping. One might expect Lagrangian velocity correlations to oscillate around zero in a vortex, so that the time integral of the Lagrangian correlations are themselves oscillating and decreasing functions of time, thus leading to severe depletion of turbulent diffusion by direct application of Taylor's integral formula. In this paper, Lagrangian velocity correlations also oscillate in some or all directions as a result of rotation alone or of combined rotation and stratification. We therefore need to calculate these oscillations in order to predict turbulent diffusion and its potential depletion in particular directions. In high-Reynolds-number turbulent flows it is usually the Eulerian velocity statistics that are (more easily) measured experimentally, not the Lagrangian ones, and it is therefore necessary to try to relate the available Eulerian correlations to the desired Lagrangian ones.

Kaneda & Ishida (2000) and Kaneda (2000) calculate these two-time Eulerian correlations using rapid distortion theory (RDT), which is justifiable in the limits of low Froude and Rossby numbers (except for the quasi-geostrophic part of the horizontal motion, for which the nonlinearity does not scale with Rossby or Froude numbers, see Godeferd & Cambon 1994 and Cambon 2001), and use a simplified form of Corrsin's conjecture to derive two-time Lagrangian correlations. The validity of RDT and the present KS is theoretically limited to times before the appearance

† In this paper particle is synonymous with fluid element.

of nonlinear dynamics. Also, polarized vortex tubes and sheets, which are not included in the RDT and KS modelling approaches presented here, might have additional effects on turbulent diffusion which are beyond our reach. Except for DNS results, we demonstrate in this paper how an unstructured velocity field is capable of anisotropically dispersing particles, by considering linear dynamics only. Thorough investigation of how nonlinear velocity field dynamics and polarized flow structures can affect Lagrangian statistics is left for future work.

DNS calculations by Kimura & Herring (1996, 1999) Kaneda & Ishida (2000) confirm the vertical capping of one-particle turbulent diffusion and Nicolleau & Vassilicos (2000, 2002) explain it in terms of energy conservation without recourse to Corrsin's conjecture. Is there a real need, therefore, for Corrsin's conjecture in the context of rapidly rotating and/or stably stratified turbulence? And is Corrsin's conjecture valid in this context? In this paper we attempt to answer these questions by use of theoretical arguments, RDT, KS and DNS, and with particular emphasis on the case of rapid rotation without stratification which has been neglected in the previous studies and which is the one case where conservation of energy arguments are not conclusive with regard to turbulent diffusion. Corrsin's conjecture concerns the relation between Eulerian and Lagrangian turbulence statistics which is of central and pivotal importance to turbulent diffusion theory. We study the validity of a simplified version of Corrsin's conjecture (which we call the 'simplified Corrsin hypothesis') and its implications for one-particle turbulent diffusion in terms of Taylor's relation (2.2) in all directions, both parallel and normal to the direction of the rotation vector.

The paper is organized as follows. In § 2 we introduce Corrsin's conjecture and the simplified Corrsin hypothesis. In § 3 we introduce the governing equations and the analytical and numerical tools used in this study: RDT, KS and DNS. In § 4 the second-order Eulerian velocity correlations are calculated on the basis of RDT solutions, and we use the simplified Corrsin conjecture introduced by Kaneda & Ishida (2000) to predict Lagrangian velocity correlation functions. Also, in this section, we show how the dispersivity of inertial waves modulates, and in fact can even in some cases prevent, the depletion of turbulent diffusion by the oscillations in the flow. At the end of § 4 we use Taylor's (1921) relation between one-particle second-order statistics and two-time Lagrangian correlations to calculate turbulent diffusivities. The validity of the simplified Corrsin conjecture is tested against KS and DNS in § 5 and in § 6 we compare our turbulent-diffusion theoretical predictions with KS and DNS results obtained at low Rossby and Froude numbers. Particular attention is given to the case of rapidly rotating turbulence without stratification, which has been neglected in the literature, and for which our results are new and perhaps the most surprising. We conclude in § 7.

2. Turbulent diffusion and Corrsin's conjecture

The position of a particle advected by a velocity field $\boldsymbol{u}$ is given by

$$\dot{\boldsymbol{x}} = \boldsymbol{v},$$

where the overdot denotes the substantial derivative following trajectories labelled by the initial ($t = 0$) position $\boldsymbol{X}$,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t),$$

and $\boldsymbol{v}(t)$ is the Lagrangian velocity related to the Eulerian velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ by

$$\boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X}, t), t). \tag{2.1}$$

One-particle dispersion in the vertical and horizontal directions is given by $\langle (x_3 - X_3)^2 \rangle$ and $\langle (x_1 - X_1)^2 + (x_2 - X_2)^2 \rangle$ respectively.

Crucial two-time quantities for turbulent diffusion are the covariances $\Delta_{ij}(t, t') = \langle \Delta x_i(t, t') \Delta x_j(t, t') \rangle$ of the displacement

$$\Delta \boldsymbol{x}(t, t') = \boldsymbol{x}(t) - \boldsymbol{x}(t') = \int_{t'}^{t} \dot{\boldsymbol{x}}(s)\, \mathrm{d}s.$$

Taylor (1921) obtained the well-known relation

$$\Delta_{ij}(t, t') = \int_{t'}^{t} \mathrm{d}s \int_{t'}^{t} \mathrm{d}s' \langle v_i(s) v_j(s') \rangle, \tag{2.2}$$

where $\langle v_i(s) v_j(s') \rangle$ denotes Lagrangian velocity correlations, which differ *a priori* from their Eulerian counterparts. Among these Lagrangian integral quantities, Δ_{33} and $\Delta_{11} + \Delta_{22}$ are particularly informative as they are the mean square of the lengths of particle excursions in the vertical and in the horizontal directions. Let us just recall here that in isotropic turbulence without rotation or stratification, $\Delta_{ii}(t, 0)$ is proportional to t^2 for short times, in agreement with a *ballistic* regime, and evolves proportionally to t for larger times, in agreement with a *Brownian* regime (Taylor 1921).

Taylor's relation (2.2) means that one-particle turbulent diffusion can be calculated provided that Lagrangian velocity statistics are known. However, it is the Eulerian velocity statistics that are (more easily) usually measured experimentally in high-Reynolds-number turbulent flows, not the Lagrangian ones, and one is therefore confronted with the task of deriving Lagrangian velocity statistics from Eulerian velocity statistics. This is a highly non-trivial task and a central problem of turbulent diffusion which is usually overcome by introducing conjectures and approximations. Corrsin's conjecture is one important such conjecture which is closely related to Kraichnan's direct interaction approximation (see Kraichnan 1970, 1977) and which has recently been used by Kaneda & Ishida (2000) to calculate one-particle turbulent diffusion in strongly stratified turbulence. We now describe this conjecture.

Equation (2.1) can also be written

$$\boldsymbol{v}(t) = \int \mathrm{d}^3\boldsymbol{y}\, \boldsymbol{u}(\boldsymbol{y}, t) \delta(\boldsymbol{y} - \boldsymbol{x}(\boldsymbol{X}, t)),$$

and the Lagrangian two-time velocity correlation needed in (2.2) is therefore given by

$$\langle v_i(t) v_j(t') \rangle = \int \mathrm{d}^3\boldsymbol{y} \langle u_i(\boldsymbol{x}(\boldsymbol{X}, t), t) u_j(\boldsymbol{y}, t') \delta(\boldsymbol{y} - \boldsymbol{x}(\boldsymbol{X}, t')) \rangle. \tag{2.3}$$

Corrsin's conjecture (Corrsin 1963) states that, for $|t - t'|$ large enough, (2.3) can be approximated by

$$\langle v_i(t) v_j(t') \rangle = \int \mathrm{d}^3\boldsymbol{y} \langle u_i(\boldsymbol{x}(\boldsymbol{X}, t), t) u_j(\boldsymbol{y}, t') \rangle \langle \delta(\boldsymbol{y} - \boldsymbol{x}(\boldsymbol{X}, t')) \rangle. \tag{2.4}$$

The reasoning behind this approximation is that $\boldsymbol{x}(\boldsymbol{X}, t')$ and $\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X}, t), t)$ are effectively uncorrelated when $|t - t'|$ is large enough. Note how this approximation equates the two-time Lagrangian velocity correlation to a weighted integral over the two-point and two-time Eulerian velocity correlation. Fourier transformation of the right-hand side of (2.4) (assuming the turbulence is statistically homogeneous) leads to

$$\langle v_i(t)v_j(t')\rangle = \int \mathrm{d}^3\boldsymbol{k}\,\hat{R}_{ij}(\boldsymbol{k};t,t')\left\langle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(x(\boldsymbol{X},t')-x(\boldsymbol{X},t))}\right\rangle \tag{2.5}$$

where

$$\hat{R}_{ij}(\boldsymbol{k};t,t') = (2\pi)^{-3}\int \langle u_i(\boldsymbol{x}+\boldsymbol{r},t)u_j(\boldsymbol{x},t')\rangle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r}.$$

Equations (2.4) and (2.5) are two different forms of Corrsin's conjecture neither of which can be of use in practice for calculations concerned with fully turbulent flows. We therefore do not directly study the validity of these equations but of a further simplification which has been introduced by Kaneda & Ishida (2000) and used by them to calculate one-particle vertical diffusion in strongly stratified turbulence. This simplification replaces (2.5) by

$$\langle v_i(t)v_j(t')\rangle = \int \mathrm{d}^3\boldsymbol{k}\,\hat{R}_{ij}(\boldsymbol{k};t,t') = \langle u_i(\boldsymbol{x},t)u_j(\boldsymbol{x},t')\rangle, \tag{2.6}$$

and we refer to it as the 'simplified Corrsin hypothesis' even though only (2.5) is due to Corrsin.

There is more than one way to derive (2.6) from (2.5) and one series of assumptions is given in Kaneda & Ishida (2000). Effectively, because Eulerian velocity correlations must be dominated by large 'eddies' (i.e. small wavenumbers $|\boldsymbol{k}|$), one might expect the high-wavenumber part of the integral (2.5) to make a negligible contribution to the right-hand side of (2.5). The term $\langle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(x(\boldsymbol{X},t')-x(\boldsymbol{X},t))}\rangle$ is equal to 1 at $\boldsymbol{k}=\boldsymbol{0}$ and might be expected to remain close to 1 at small wavenumbers and decrease with increasing wavenumber. This term might therefore be close to 1 in the range of wavenumbers that makes the dominant contribution to the integral in (2.5); on replacing it with 1 in (2.5) one obtains (2.6) without expecting much error, independently of the exact form of the energy spectrum, as long as a logarithmic slope steeper than -1 is assumed.

Corrsin's conjecture (2.5) has been tested against kinematic simulations of two- and three-dimensional flows with an energy spectrum sharply peaked about one well-defined lengthscale (Kraichnan 1970, 1977; Lundgren & Pointin 1976) with the conclusion that it is valid for all times (not only large times) provided that there is no helicity and that the flow is not frozen in time. Furthermore, as stated by Kraichnan (1970), "if the correlation time of the Eulerian velocity field is very short compared with the eddy circulation time, the Eulerian and Lagrangian velocity covariances are almost the same", i.e. (2.6) holds. Is this statement true in turbulent-like (KS) velocity fields with a wide range of excited wavenumbers and correlation times comparable with eddy-circulation times? And what happens when stratification and/or rotation introduce their own timescales? How do predictions of one-particle dispersion based on (2.2), (2.6) and RDT calculations of two-time Eulerian velocity correlations (as in Kaneda & Ishida 2000) compare with numerical simulations (KS and DNS) of rapidly rotating turbulence with and without stable stratification? These questions are pertinent in particular because such velocity fields are more representative of real turbulence than those considered by Kraichnan (1970, 1977). We now describe the methods we use to obtain these velocity fields: DNS, RDT and KS. The DNS and KS velocity fields are used to integrate particle trajectories and the RDT solutions of the linearized dynamical equations are used both in KS and for analytical calculations of Eulerian two-time velocity correlations.

3. Background equations and methods

The Navier–Stokes equations using the Boussinesq approximation in a rotating frame are

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + 2\Omega\boldsymbol{n}\times\boldsymbol{u} + \nabla p - \nu\nabla^2\boldsymbol{u} = \boldsymbol{b}, \tag{3.1}$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)b - \chi\nabla^2 b = -N^2\boldsymbol{n}\cdot\boldsymbol{u}, \tag{3.2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{3.3}$$

for the fluctuating velocity $\boldsymbol{u}$, the pressure p divided by a mean reference density, and the buoyancy force $\boldsymbol{b}$; ν and χ are the kinematic viscosity and thermal diffusivity respectively. The vector $\boldsymbol{n}$ denotes the vertical unit upward direction aligned with both the gravitational acceleration $\boldsymbol{g} = -g\boldsymbol{n}$ and the angular velocity of the rotating frame $\boldsymbol{\Omega} = \Omega\boldsymbol{n}$. The buoyancy force is related to the fluctuating temperature field τ by $\boldsymbol{b} = -\boldsymbol{g}\beta\tau = b\boldsymbol{n}$, through the coefficient of thermal expansivity β. With temperature stratification characterized by the vertical gradient γ, the Brunt–Väisälä frequency $N = \sqrt{\beta g\gamma}$ appears as the characteristic frequency of buoyancy–stratification. Hence the linear operators in equations (3.1) and (3.2) display the two frequencies N and 2Ω, and we define $B = 2\Omega/N$ as a measure of the relative importance of rotation and stratification. Without loss of generality the fixed frame of reference is chosen such that $n_i = \delta_{i3}$. Therefore, u_3 is the vertical velocity component.

3.1. *Direct numerical simulation*

Using DNS, the system of equations (3.1), (3.2), (3.3) is solved by a numerical spectral collocation method – with 192 Fourier polynomials in each direction of space – in a now classical way for homogeneous turbulence simulations (see e.g. Vincent & Meneguzzi 1991). Prior to solving, the system of equations is rewritten using the vector identity

$$(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{u} + \frac{\nabla|\boldsymbol{u}^2|}{2},$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the vorticity. The background rotation 2Ω can therefore be directly added to the vertical vorticity. The boundary conditions are periodic in all three directions, and the numerical domain is a cubic box of side L_0, so that the dimensionless wave vector components are $k_i^* = k_i 2\pi/L_0$. Spatial derivatives are treated numerically in Fourier space using the pseudo-spectral method and aliasing is treated in spectral space by a spherical truncation of the Fourier components of the fluctuating velocity field, using a 2/3 de-aliasing rule at every time step. The time scheme is third-order Adams–Bashforth, and the viscous term is integrated exactly using the change of variable $\boldsymbol{u}'(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k})\exp(\nu k^2 t)$.

A total of 1000 fluid particles are located initially with a uniform random distribution in the numerical box, and are followed by solving the Lagrangian trajectory equation $\mathrm{d}(\boldsymbol{x}(\boldsymbol{X},t))/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t),t)$ at each timestep with a second-order Runge–Kutta scheme. The necessary interpolation for obtaining the velocity $\boldsymbol{u}(\boldsymbol{x}(\boldsymbol{X},t))$ at the location of the particle is performed with great accuracy by a Lagrange interpolation scheme which makes use of six points in each direction of space.

The Eulerian velocity field is initialized using an isotropic field obtained from a pre-computation of (3.1) with zero buoyancy force and zero angular velocity but with large-scale forcing in order to achieve a classical DNS isotropic turbulence stationary state. This pre-computation is itself initialized by a velocity field with uniformly distributed Fourier phases and an analytically prescribed energy spectrum E^0 as in Orszag & Patterson (1972). The peak of this spectrum is chosen at $k_p^* = 4.52$, and

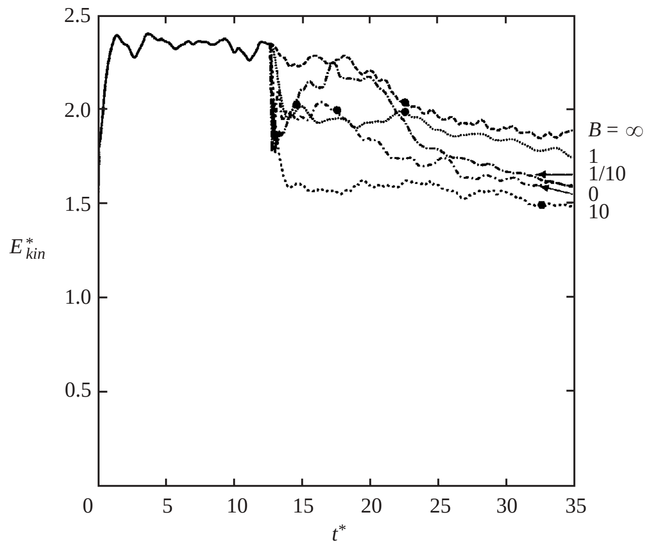

FIGURE 1. Time evolution of the dimensionless kinetic energy for the entire simulation time span, for the presented DNS runs. $t^* = t/\tau$ is the dimensionless time since the beginning of the isotropic pre-computation, drawn as a solid line ($\tau = 1.01$ is the eddy turnover time at the end of the pre-computation). The anisotropic runs are initiated at $t^* = 12.59$. All the values of B are plotted, $B = 0, 1/10, 1, 10, \infty$. Black circles indicate the end of computions of Lagrangian statistics, corresponding to different times when non-dimensionalized by the linear timescale (either $2\pi/N$ or π/Ω).

the corresponding (non-dimensional) total kinetic energy $E^*_{kin} = E_{kin}/u'^2$ is 1.5 at $t = 0$ ($u' = \sqrt{2E_{kin}/3}$ being the r.m.s. velocity scale in one direction). The pre-computation of forced isotropic turbulence is performed for a dimensionless time $t/\tau = 12.59$, allowing the turbulent dynamics to set up a realistic distribution with built-up triple correlations (τ is the eddy turnover time at the end of the isotropic run). The evolution of the kinetic energy during this stage is shown in figure 1. During this period and after, the Eulerian velocity field is forced by maintaining the large scales – small wavenumbers – at their level at $t = 0$. This is achieved by multiplying, at every timestep, each spectral coefficient $\hat{u}(\boldsymbol{k}^*)$ within the spherical spectral shell $|\boldsymbol{k}^*| < 4.52$ by $E^0(k^*)/|\hat{u}(\boldsymbol{k}^*)|$. In so doing, one only modifies the amplitude of the Fourier modes concerned, letting their phase evolve freely through the action of (3.1), (3.2) and (3.3), so as to minimize the impact of forcing on anisotropy, which we want to develop as 'naturally' as possible. This simple forcing method yields turbulence statistics that do not depart too much from those at the initial time (i.e. $t = 12.59\tau$), as would occur for a freely decaying run. The turbulent flow is nevertheless unsteady (see figure 1). The Lagrangian statistics are computed starting at $t = 12.59\tau$, that is fluid particles are released at this instant only, and from now on in this paper we use the notation t for $t - 12.59\tau$, so that the initial time of fluid particle release is $t = 0$. Eulerian velocity correlations are obtained for $t \geqslant 0$ by spatial averages over all the collocation points. The initial ($t = 0$ with the new origin) values of the non-dimensional numbers are as follows: the Reynolds number $Re = u'L/\nu$ at $t = 0$ is the same for all the runs, $Re(0) = 217$, where $u' = \sqrt{2E_{kin}/3}$ is the r.m.s. velocity scale and L the integral lengthscale. The Froude number is therefore computed as $Fr = u'/(NL)$ and the Rossby number as $Ro = u'/(2\Omega L)$. Their initial respective values for the different runs are shown in table 1.

B	∞	10	1	1/10	0
$Ro(0)$	0.81	0.162	0.81	1.61	∞
$Fr(0)$	∞	1.62	0.81	0.16	0.4

Table 1. Initial Rossby and Froude numbers for DNS runs at different values of parameter B.

3.2. *Linearized dynamical equations*

As in RDT, equations (3.1), (3.2) can be linearized for small enough Rossby and Froude numbers. Solutions are easily found in terms of plane waves, and consist of a superposition of steady and oscillating modes of motion, the latter corresponding to dispersive, inertia–gravity, waves (see Cambon 2001). Fourier space is useful for taking into account the divergence-free constraint (3.3), and for allowing exact treatment of the pressure gradient in (3.1). This is important because pressure disturbances are responsible for a coupling of vertical and horizontal velocity components, and for the anisotropic dispersion law of the wavy part of the flow. The solenoidal part of the horizontal field is unaffected by the wave part of the flow and remains constant in the linear inviscid limit: it corresponds to the 'vortex' part of the flow in the purely stratified case (see Riley, Metcalfe & Weissman 1981). Accordingly, complete linear solutions (which we call RDT solutions) for the velocity in terms of plane waves may be written as

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})[\boldsymbol{A}^0(\boldsymbol{k}) + \boldsymbol{A}^1(\boldsymbol{k})\exp(\mathrm{i}\sigma t) + \boldsymbol{A}^{-1}(\boldsymbol{k})\exp(-\mathrm{i}\sigma t)] \tag{3.4}$$

(and similarly for b) where the three vectors $\boldsymbol{A}^\epsilon(\boldsymbol{k})$, $\epsilon = 0, \pm 1$ correspond to the one steady and two wavy modes (see §4 for details) and are such that $\boldsymbol{A}^\epsilon(\boldsymbol{k})\cdot\boldsymbol{k} = 0$ for incompressibility. The unsigned dispersion frequency σ of inertia–gravity waves is given by

$$\sigma = \sqrt{N^2 \sin^2\theta + 4\Omega^2\cos^2\theta},$$

where $\theta(\boldsymbol{k})$ is the angle between $\boldsymbol{k}$ and the vertical direction. As a consequence of our use of Fourier space and of $\boldsymbol{A}^\epsilon(\boldsymbol{k})\cdot\boldsymbol{k} = 0$, the five unknowns u_1, u_2, u_3, p, b in physical space are now reduced to three in Fourier space, namely two velocity components which we detail in §4 and the Fourier transform of b.

Note that two particular cases require special treatment. For pure rotation $N = 0$ the vector $\boldsymbol{A}^0$ corresponding to the steady part of the flow vanishes in (3.4). And in the case $B = 1$, it is the dependence of the dispersion frequency on $\boldsymbol{k}$ that vanishes, so that the group velocity is zero for all wavevectors. (This special character of the $B = 1$ case has also been noted in DNS by Iida & Nagano 1999 and in RDT by Kaneda 2000 and Hanazaki 2002.)

3.3. *Kinematic simulation*

KS of strongly stratified and rotating non-decaying turbulence has been introduced by Nicolleau & Vassilicos (2000, 2003), where detailed presentations can be found. This KS is based on the velocity field (3.4) which incorporates the linear dynamics. Similarly to initializations of pseudo-spectral DNS, the velocity field at time $t = 0$ is isotropic, incompressible and has a prescribed Eulerian energy spectrum $E(k)$ (this spectrum is chosen with an inertial range, as for developed turbulence). Moreover, the incompressibility and isotropy of initial conditions constrain the angular dependences of the vectors $\boldsymbol{A}^{\epsilon}(\boldsymbol{k})$ and $\boldsymbol{k}$. (The link of $\boldsymbol{A}^0$, $\boldsymbol{A}^1$ and $\boldsymbol{A}^{-1}$ to the initial realization is fully specified by equations (4.3)–(4.5), as explained at the end of §4.1.) At given parameter B, the initial energy spectrum determines the average amplitudes of the vectors $\boldsymbol{A}^{\epsilon}(\boldsymbol{k})$, where these averages are taken over many realizations of the velocity field. Different realizations are obtained by specifying different angular dependences of the vectors $\boldsymbol{A}^{\epsilon}(\boldsymbol{k})$ and $\boldsymbol{k}$. Great care has to be taken for the angular distribution of modes in these realizations, for the phase and amplitude of oscillation of turbulence to be properly reproduced.

Following Nicolleau & Vassilicos (2000, 2002), additional time decorrelation terms are included in the KS velocity field to yield

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + \mathrm{i}\omega t)[\boldsymbol{A}^0(\boldsymbol{k}) + \boldsymbol{A}^1(\boldsymbol{k})\exp(\mathrm{i}\sigma t) + \boldsymbol{A}^{-1}(\boldsymbol{k})\exp(-\mathrm{i}\sigma t)], \tag{3.5}$$

where the frequencies $\omega(k) = \lambda\sqrt{k^3E(k)}$ with $k = |\boldsymbol{k}|$ and in this paper we investigate both options, $\lambda = 1$ and $\lambda = 0$. The inclusion of these time decorrelation terms is intended to model the Eulerian decorrelating effect of the nonlinear advection which we have neglected in the RDT solutions. Indeed, similarly to Nicolleau & Vassilicos (2000) in the purely stratified case, we report in §5 of this paper that the inclusion of the frequencies $\omega(k)$ accelerates the decay of some Eulerian velocity correlation functions obtained by KS of strongly stratified and rotating turbulence, thus leading to more realistic Eulerian correlations. The impact on Lagrangian correlations is shown to be unimportant.

The number of wavevectors over which the summation in every realization of the velocity field (3.5) is carried out is 2000 to 6000, which is much less than in a 192^3 DNS. It is therefore practicable to average over many realizations of the flow in KS, whereas the high cost involved in DNS allows the use of only one DNS realization in practice. Accordingly, KS achieves better converged statistics than DNS. It should be stressed that KS is a Lagrangian model of turbulent diffusion and should therefore be used to integrate particle trajectories and generate Lagrangian statistics. Its Eulerian flow structure is flawed as it does not incorporate the persistent characteristic pancake- or cigar-shaped structures that DNS can capture. Of course, the results should coincide with those of analytical RDT when calculating Eulerian velocity correlations, this being an unavoidable validation test.

In our KS, in accordance with Nicolleau & Vassilicos (2000), particles are released at a random time large enough for the velocities to have reached their asymptotic r.m.s. values. This is consistent with KS as a Lagrangian model. Here in contrast to DNS, averages are taken over realizations and in such an approach there is no reason why particles should be released at the same time in each realization.

4. Analytical calculations using RDT

4.1. *Projection of the fluctuating fields onto the basis of eigenmodes*

The notation and equations in this subsection are the same as in Cambon (2001), which have much in common with Kaneda (2000) (see also Godeferd & Cambon 1994 for details of the mathematical formulation in the stratified case).

In order to unify the mathematical formulation, we shall use a vector $\hat{\boldsymbol{w}}$ whose first two components are $\hat{u}_1$ and $\hat{u}_2$, the components of the spectral coefficient $\hat{\boldsymbol{u}}(\boldsymbol{k})$ of $\boldsymbol{u}$ in the plane orthogonal to the wavenumber $\boldsymbol{k}$. The *local frame* in this plane is chosen such that $\hat{u}_1$ is the component in the horizontal plane. The corresponding local frame is the Craya–Herring frame described at length in the Appendix. The last component

$\hat{w}_3$ contains the buoyancy force scaled as a velocity:

$$\hat{w}_3 = \mathrm{i}N^{-1}\hat{b}.$$

In the local frame, the linearized system of equations (3.1)–(3.3) becomes

$$\partial_t \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \mathrm{i}\hat{w}_3 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_r & 0 \\ \sigma_r & 0 & -\sigma_s \\ 0 & \sigma_s & 0 \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \mathrm{i}\hat{w}_3 \end{pmatrix} = \mathbf{0} \tag{4.1}$$

or $\partial_t \hat{\boldsymbol{w}} + \boldsymbol{M}\hat{\boldsymbol{w}} = 0$, where

$$\sigma_r = 2\Omega\cos\theta, \quad \sigma_s = N\sin\theta, \quad \sigma = \sqrt{\sigma_r^2 + \sigma_s^2} \tag{4.2}$$

are respectively the unsigned dispersion-law frequencies for inertial waves, gravity waves, and inertia–gravity waves.

The system of equations (4.1) is easily solved by diagonalizing the matrix $\mathbf{M}$. Three eigenmodes are obtained *in the local frame* as follows:

$$\boldsymbol{N}^0 = \begin{pmatrix} \sigma_s/\sigma \\ 0 \\ \sigma_r/\sigma \end{pmatrix}, \quad \boldsymbol{N}^1 = \frac{\sqrt{2}}{2}\begin{pmatrix} -\sigma_r/\sigma \\ \mathrm{i} \\ \sigma_s/\sigma \end{pmatrix}, \quad \boldsymbol{N}^{-1} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\sigma_r/\sigma \\ -\mathrm{i} \\ \sigma_s/\sigma \end{pmatrix}$$

They are related to the eigenvalues 0, σ and $-\sigma$, respectively. $\boldsymbol{N}^0$ is the stationary mode, which coincides with the quasi-geostrophic mode (QG). Its two components are the horizontal divergence-free part of the velocity, or vortex mode, and one associated with the temperature field. $\boldsymbol{N}^{\pm 1}$ are the wave modes, also called ageostrophic modes hereinafter in agreement with classic geophysical literature. The wave–vortex terminology was used by Riley *et al.* (1981) for analysing DNS of stably stratified turbulence, whereas geostrophic and ageostrophic were terms used by Bartello (1995) and Babin, Mahalov & Nicolaenko (1998).

The orthonormal basis of eigenmodes is used to express $\hat{\boldsymbol{w}}$ as

$$\hat{\boldsymbol{w}} = \sum_{\epsilon=0,\pm 1} \xi^{\epsilon}\boldsymbol{N}^{\epsilon},$$

with

$$\xi^{\epsilon} = \hat{\boldsymbol{w}} \cdot \boldsymbol{N}^{-\epsilon}, \quad \epsilon = 0, \pm 1.$$

Accordingly, the linearized problem (4.1) obtained from (3.1)–(3.3) is writen $(\partial_t + \mathrm{i}\epsilon\sigma)\xi_\epsilon = 0$, whose general solution, given initial values $\hat{\boldsymbol{w}}(\boldsymbol{k}, 0)$, becomes

$$\hat{\boldsymbol{w}}(\boldsymbol{k}, t) = \sum_{\epsilon=0,\pm 1} \boldsymbol{N}^{\epsilon}\mathrm{e}^{-\mathrm{i}\epsilon\sigma t}[\boldsymbol{N}^{-\epsilon} \cdot \hat{\boldsymbol{w}}(\boldsymbol{k}, 0)]. \tag{4.3}$$

At this point, one may compute the $\boldsymbol{A}^{\epsilon}$ defined in (3.4), since they are directly related to the initial conditions through the two kinematic components $A_i^{\epsilon} = N_i^{\epsilon}(\boldsymbol{N}^{-\epsilon} \cdot \hat{\boldsymbol{w}}(\boldsymbol{k}, 0))$, $i = 1, 2$.

4.2. *RDT solutions for second-order Eulerian statistics*

From equation (4.3), the linear solution for any statistical Eulerian quantity is readily derived. Second-order statistics are defined through the spectrum of two-point correlations V_{ij}:

$$\tfrac{1}{2}\langle \hat{w}_i^*(\boldsymbol{p}, t)\hat{w}_j(\boldsymbol{k}, t')\rangle = V_{ij}(\boldsymbol{k}, t, t')\delta(\boldsymbol{k} - \boldsymbol{p}). \tag{4.4}$$

The RDT equations for V_{ij} are unchanged if one accepts the 'stretching' of notation $V_{ij} = \langle \hat{w}_i^*(\boldsymbol{k}, t)\hat{w}_j(\boldsymbol{k}, t')\rangle$ (instead of the exact equation (4.4)) for linking V_{ij} to its initial value through the general solution (4.3).

Its components $V_{11}(\boldsymbol{k}, t, t' = t)$, $V_{22}(\boldsymbol{k}, t, t' = t)$, $V_{33}(\boldsymbol{k}, t, t' = t)$, $V_{23}(\boldsymbol{k}, t, t' = t)$ are therefore spectral densities of the vortex energy, kinetic wave energy, potential energy, and vertical buoyancy flux, in the absence of system rotation. Only the components with indices 1 and 2 contribute to the spectral tensor $\hat{R}_{ij}(\boldsymbol{k}, t, t')$ which appears in (2.5).

Without any calculation, it is obvious that the RDT history of any two-point or single-point correlation can be obtained in a quasi-analytical way provided initial data have simple two-point statistics. For instance, initial three-dimensional isotropic conditions can be chosen as

$$V_{11} = V_{22} = \frac{E_c(k)}{8\pi k^2}, \quad V_{33} = \frac{E_p(k)}{8\pi k^2}, \tag{4.5}$$

at $t = t' = 0$, all other cospectra being zero. Furthermore, the spherically averaged energy spectra can be chosen proportional, such that $E_p(k) = \alpha E_c(k)$, for even greater simplicity. Initial data in KS and DNS are restricted to $\alpha = 0$ in this article, as in Nicolleau & Vassilicos (2000), but RDT calculations can be easily carried out for any arbitrary value of α, and the particular case $\alpha = 1$ deserves further investigations. (Non-zero values for α were also considered in RDT by Hanazaki & Hunt 1996.) Using the above initial data, the spectra of Eulerian correlations $N^{-2}\langle b(t')b(t)\rangle$, $\langle u_3(t')u_3(t)\rangle$, and $\langle u_i(t')u_i(t)\rangle$ are calculated from linear combinations of the solution of the V_{ij} integrated over $\boldsymbol{k}$ (see the Appendix). One finds

$$\begin{aligned}
&\langle u_i(t')u_i(t)\rangle + N^{-2}\langle b(t')b(t)\rangle \\
&\quad = E_{kin}(0)\int_0^1 \left[(1-(1-\alpha)\frac{\sigma_r^2}{\sigma^2} + \left(2-(1-\alpha)\frac{\sigma_s^2}{\sigma^2}\right)\cos\sigma(t-t')\right]\mathrm{d}\mu \\
&\langle u_3(t')u_3(t)\rangle \\
&\quad = E_{kin}(0)\int_0^1 \frac{(1-\mu^2)}{\sigma^2}\left[\left(\sigma_r^2+\alpha\sigma_s^2\right)\sin(\sigma t)\sin(\sigma t') + \sigma^2\cos(\sigma t)\cos(\sigma t')\right]\mathrm{d}\mu
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\langle u_1(t')u_1(t)\rangle + \langle u_2(t')u_2(t)\rangle = E_{kin}(0)\int_0^1 &[F(B,\mu) + G(B,\mu)(\cos\sigma t + \cos\sigma t') \\
&+ C(B,\mu)\cos\sigma t\cos\sigma t' + D(B,\mu)\sin\sigma t\sin\sigma t']\,\mathrm{d}\mu,
\end{aligned} \tag{4.7}$$

with

$$\left.\begin{aligned}
F(B,\mu) = \sigma_s^2\left(\sigma_s^2+\alpha\sigma_r^2\right)/\sigma^4, \quad G(B,\mu) = (1-\alpha)\sigma_r^2\sigma_s^2/\sigma^4, \\
C(B,\mu) = \sigma_r^2\left(\sigma_r^2+\alpha\sigma_s^2\right)/\sigma^4 + \mu^2, \quad D(B,\mu) = \left[\sigma_r^2(1+\mu^2) + \alpha\sigma_s^2\mu^2\right]/\sigma^2,
\end{aligned}\right\} \tag{4.8}$$

and with σ_r, σ_s and σ given by (4.2) and $\mu = \cos\theta$. The initial amount of kinetic energy in the whole flow is $E_{kin}(0)$. It is clear from equations (4.6) to (4.8) that the oscillatory behaviour of Eulerian velocity correlations is modulated by integrations over the wave dispersivities $\sigma_r = \sigma_r(\mu)$, $\sigma_s = \sigma_s(\mu)$ and $\sigma = \sigma(\mu)$.

For pure rotation, $N = 0$, equations (4.6) and (4.7) reduce to

$$\langle u_3(t')u_3(t)\rangle = E_{kin}(0)\int_0^1 [(1-\mu^2)\cos 2\Omega\mu(t-t')]\,\mathrm{d}\mu$$

and

$$\langle u_1(t')u_1(t)\rangle + \langle u_2(t')u_2(t)\rangle = E_{kin}(0)\int_0^1 [(1+\mu^2)\cos 2\Omega\mu(t-t')]\,\mathrm{d}\mu.$$

As for (4.6) and (4.7) the integrands in the above equations are simplified forms of $V_{22}(1-\mu^2)$ and $V_{11}+\mu^2 V_{22}$ given in the Appendix. Note from the above equations the important conclusion that both vertical and horizontal Eulerian velocity two-time correlations decay to zero as a result of wave dispersivity.

For pure rotation, setting $t = t'$ in the above equations we retrieve the well-known result (also reflected by the measure of Reynolds stress anisotropy) that the initial isotropy of the velocity field is conserved by RDT in the sense that $\langle u_3^2(t)\rangle/[(\langle u_1^2(t)+u_2^2(t)\rangle)/2] = 1$.

4.3. *Analytical approach to turbulent diffusion*

Only the case of pure rotation is treated in detail in this subsection. Analytical results for the general case, $N \neq 0$, will be given in the next two sections, where numerical results are discussed.

Making use of the simplified Corrsin hypothesis (2.6), and therefore equating RDT Eulerian and Lagrangian velocity correlations in Taylor's equation (2.2), yields

$$\Delta_{33}(s,0) = \frac{E_{kin}(0)}{2\Omega^2}\int_0^1 (1-\mu^2)\frac{1-\cos 2\Omega\mu s}{\mu^2}\,\mathrm{d}\mu \tag{4.9}$$

and

$$\Delta_{11}(s,0)+\Delta_{22}(s,0) = \frac{E_{kin}(0)}{2\Omega^2}\int_0^1 (1+\mu^2)\frac{1-\cos 2\Omega\mu s}{\mu^2}\,\mathrm{d}\mu, \tag{4.10}$$

where use has been made of

$$\int_0^s \mathrm{d}t \int_0^s \mathrm{d}t'(\cos 2\Omega\mu t\cos 2\Omega\mu t' + \sin 2\Omega\mu t\sin 2\Omega\mu t') = (1-\cos 2\Omega\mu s)/(2\Omega^2\mu^2).$$

The integrals (4.9) and (4.10) result in analytical solutions as follows:†

$$\left\langle \tilde{x}_3^2(s)\right\rangle = \Delta_{33}(s,0) = \frac{E_{kin}(0)}{4\Omega^2}\left[4Si(2\Omega s)\Omega s + 2\cos 2\Omega s - 4 + \frac{\sin 2\Omega s}{\Omega s}\right] \tag{4.11}$$

and

$$\left\langle \tilde{x}_1^2(s)+\tilde{x}_2^2(s)\right\rangle = \Delta_{11}(s,0)+\Delta_{22}(s,0) = \frac{E_{kin}(0)}{4\Omega^2}\left[4Si(2\Omega s)\Omega s + 2\cos 2\Omega s - \frac{\sin 2\Omega s}{\Omega s}\right] \tag{4.12}$$

with $Si(t) = \int_0^t u^{-1}\sin u\,\mathrm{d}u$.

Equations (4.11) and (4.12) show that, at large Ωt, both the vertical and horizontal r.m.s. displacements behave as $\sqrt{\Omega t}$, or more precisely

$$\left\langle \tilde{x}_3^2(t)\right\rangle \sim \left\langle \tilde{x}_1^2(t)+\tilde{x}_2^2(t)\right\rangle \sim \frac{E_{kin}(0)}{2\Omega^2}\pi\Omega\, t \tag{4.13}$$

using the limit $Si(\infty) = \pi/2$. In the limit $\Omega \to 0$, equations (4.9) and (4.10) tend to

$$\left\langle \tilde{x}_3^2(t)\right\rangle = \tfrac{1}{2}\left(\left\langle \tilde{x}_1^2(t)\right\rangle + \tilde{x}_2^2(t)\rangle\right) = \tfrac{2}{3}E_{kin}\, t^2 \tag{4.14}$$

† Integrals similar to (4.9) and (4.10) were assumed to diverge by Kaneda (2000), since he only looked at the non-oscillating part of their integrand. Convergence at $\mu = 0$ is guaranteed if the cosine term is correctly accounted for.

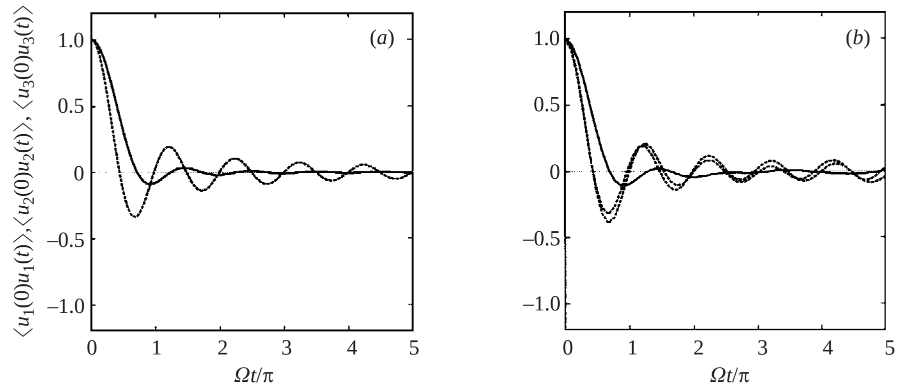

FIGURE 2. Purely rotating case $B = \infty$. (*a*) Analytical RDT for the two-time Eulerian velocity correlations: ----, horizontal correlations $\langle u_1(0)u_1(t)\rangle = \langle u_2(0)u_2(t)\rangle$; ——, vertical correlation $\langle u_3(0)u_3(t)\rangle$. (*b*) KS Eulerian correlations for $\lambda = 0$: ----, horizontal correlations $\langle u_1(0)u_1(t)\rangle$ and $\langle u_2(0)u_2(t)\rangle$ (as in the following KS plots); ——, vertical correlation $\langle u_3(0)u_3(t)\rangle$.

since $\cos 2\Omega\mu s \sim 1 - 2\Omega^2(\mu s)^2$, for all times. This ballistic behaviour is valid only at small times, thus pointing to the inadequacy of RDT and/or simplified Corrsin hypothesis in the case without rotation.

Diffusivity in purely rotating turbulence, given by (4.13), is therefore identical to the classical behaviour of isotropic turbulence, after a time large enough to reach the 'Brownian' regime for which $\langle \tilde{x}^2\rangle \propto t$. The asymptotic ratio $\langle \tilde{x}_3{}^2\rangle/\langle \tilde{x}_1{}^2\rangle$ shows that isotropy is broken, although not in a dramatic way since it tends towards 2 at large t. The depletion of turbulent diffusion expected from the oscillatory behaviour of the rotating flow is prevented in this case by the wave dispersivity, specifically by integrations over this wave dispersivity as clearly seen in equations (4.9) and (4.10). In other cases of combined rotation with stratification this wave dispersivity simply modulates, and in fact weakens, the depletion of turbulent diffusion due to flow oscillations.

5. The simplified Corrsin hypothesis: Eulerian and Lagrangian two-time correlations from RDT, KS and DNS

In this section we compare Eulerian and Lagrangian two-time correlations obtained by RDT, KS and DNS. Comparisons between Eulerian two-time correlations obtained with RDT, and KS with $\lambda = 0$, are used to validate our KS and to determine the time ranges where comparisons are possible.

5.1. *Pure rotation, $N = 0$*

Choosing $t' = 0$, the RDT solutions predict decay to zero for both horizontal and vertical Eulerian two-time velocity correlations when t increases (figure 2). Results from KS ($\lambda = 0$) compare satisfactorily, as shown on the same figure. Note, however, that a sufficiently refined KS discretization of the KS sum (3.5) is required to achieve good comparison. Damped oscillations reflect the dispersivity of inertial waves, as expected. Comparing $\langle u_3(0)u_3(t)\rangle$ with $\langle u_1(0)u_1(t)\rangle$, a departure from isotropy is observed although remaining quite small. The simplified Corrsin hypothesis (2.6) is found to be sufficiently well satisfied in KS both for $\lambda = 0$ and $\lambda = 1$ in both the horizontal and the vertical directions (see figure 3*b* for $\lambda = 0$ and 3*c* for $\lambda = 1$

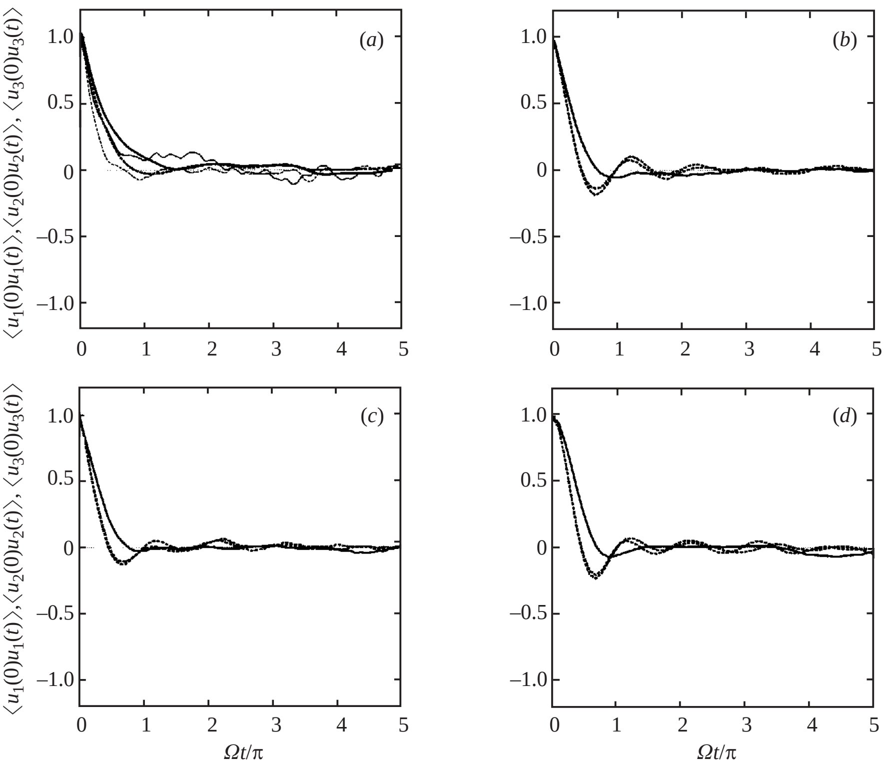

FIGURE 3. Purely rotating case $B = \infty$. ----, two-time horizontal velocity correlations; ——, vertical correlation, for (*a*) DNS results with Eulerian correlations drawn with heavy lines, and Lagrangian ones with thin lines; (*b*) KS Lagrangian correlations ($\lambda = 0$); (*c*) KS Lagrangian correlations ($\lambda = 1$); (*d*) KS Eulerian correlations with $\lambda = 1$.

and compare with figures 2*b* and 3*d* respectively). Our DNS results (see figure 3*a*) confirm the validity of the simplified Corrsin hypothesis in the horizontal direction. They are less conclusive in the vertical direction, as we can observe a rather clear difference between the Eulerian and the Lagrangian vertical correlations from about $\Omega t/\pi \simeq 1$. However, one must take into account the lack of statistical sampling for the Lagrangian velocity correlations, which are computed using 1000 fluid particles in the DNS, which is much less than the 192^3 collocation points used to calculate the Eulerian correlations. This lack of sampling causes a difference in the oscillation amplitude between horizontal Eulerian and Lagrangian correlations, and may be the cause of (or at least contribute to) the observed departure in the vertical. Moreover, no ensemble average is performed for the DNS results, so that large overall oscillations in the correlations are present, which would tend to be smoothed out when averaging with results from other runs with different initial turbulent phases.

5.2. *General case with stable stratification*

In the stratified case, with and without rotation, initial isotropy is not conserved by RDT for *single-time* Eulerian correlations, in the absence of initial equipartition, that is when $(1/2)N^{-2}\langle b^2\rangle(0) \neq E_{kin}(0)/2$ or $\alpha \neq 1$. The asymptotic value of $2\langle u_3^2\rangle/(\langle u_1^2 + u_2^2\rangle)(Nt \gg 1)$ is found to be a function of α and B. The depletion in the vertical contribution, which is found in the particular case $B = 0$, $\alpha = 0$, has nothing to do with the collapse of vertical motion, and the above ratio could be larger than 1 for different values of α and/or different values of B (see Cambon 2001).

In the general case for which N is not zero, *two-time* Eulerian correlations calculated by RDT depend on t and t' separately (for all $\alpha \neq 1$), and not only through their difference, as in the case of pure rotation and in the isotropic case. Correlations in the vertical direction decrease to zero for large Nt, whereas they tend to a non-zero plateau for the horizontal direction according to the non-zero value of $F(B, \alpha, \mu)$ in (4.7) and (4.8). The main difference with the case of pure rotation is that the horizontal two-time velocity correlation (4.7) tends to a non-zero limit as Nt increases, whereas its vertical counterpart decreases to zero. Hence, two-time Eulerian correlations become strongly anisotropic at sufficiently large separation times. The Eulerian two-time correlations obtained from RDT and KS with $\lambda = 0$ are compared in figure 4 for a choice of values of B, and the comparison is satisfactory.

The simplified Corrsin hypothesis is found to hold in KS in the vertical direction for all finite values of B and for both values of λ. This is illustrated in figures 5–8 for Eulerian and Lagrangian vertical and horizontal correlations in KS for different B values and $\lambda = 0$ or $\lambda = 1$: figure 5 for $B = 0$, figure 6 for $B = 1/10$, figure 7 for $B = 1$, figure 8 for $B = 10$. It is interesting to note that the Lagrangian two-time correlations are unaffected by the value of λ except in the vertical direction for the case $B = 1$. In the horizontal direction the simplified Corrsin hypothesis is recovered by KS only for $\lambda = 1$: in figures 3, and 5 to 8 Lagrangian and Eulerian correlations look quasi-identical, and even those in the horizontal direction have the same decay to zero for $\lambda = 1$. When $\lambda = 0$, the horizontal Eulerian two-time correlations obtained by KS and by RDT do not decay to zero, as shown in figure 4, which means that the simplified Corrsin hypothesis is not valid in that case. In the non-dispersive $B = 1$ KS results, the difference between Eulerian correlations for $\lambda = 0$ and $\lambda = 1$ is therefore striking (figures 4f and 7d). This effect of λ on the decay of Eulerian velocity correlations is also present in KS results for isotropic turbulence ($\Omega = N = 0$) as shown in figure 9. Figure 9 shows clearly that Eulerian velocity correlations are equal to 1 for all time in KS when $\lambda = 0$ but decay with time when $\lambda = 1$. From tracking of fluid particles in the laboratory, Mann & Ott (2002) have found some evidence that the simplified Corrsin hypothesis is valid in isotropic turbulence.

Our DNS results confirm, at least qualitatively, that the simplified Corrsin hypothesis is valid in the vertical direction for all values of B but seem to invalidate this hypothesis in the horizontal directions. In the stratification-dominant cases, from figures 5 and 6 the conclusion is very clear: apart from small differences due to statistical aliasing, Eulerian and Lagrangian DNS correlations are very close in the vertical direction, and also compare very well with KS evolutions. On the other hand, horizontal DNS correlations separate at about half a Brunt–Väisälä period $Nt/2\pi \simeq 0.5$, for both $B = 0$ and $B = 1/10$ cases. For the non-dispersive case (figure 7), while Eulerian and Lagrangian DNS correlations are nearly identical at the initial stage, one observes a slight separation of the horizontal correlations. For the rotation-dominant case $B = 10$, the simplified Corrsin hypothesis seems well-verified in both the vertical and the horizontal directions (figure 8).

6. Results for turbulent diffusion

6.1. *Pure rotation*

The analytical results of equations (4.11) and (4.12) for $\langle \tilde{x}_3^2 \rangle$ and $\langle \tilde{x}_1^2 \rangle$ are plotted in non-dimensional form in figure 10, along with results from KS with $\lambda = 0$ at two

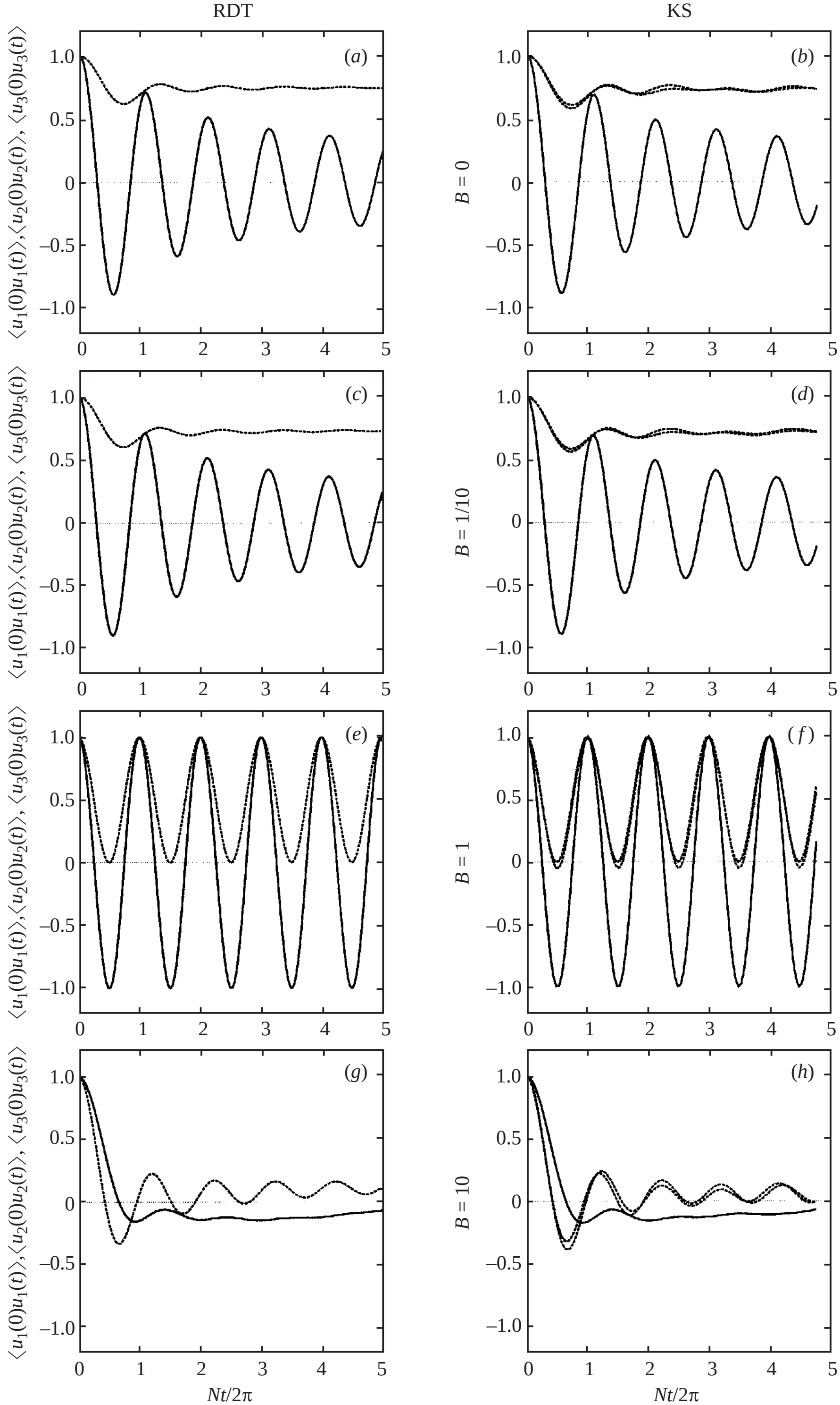

FIGURE 4. Eulerian two-time velocity correlations for $B = 0$: ----, horizontal; ——, vertical. (*a*) RDT and (*b*) KS with $\lambda = 0$; $B = 1/10$: (*c*) RDT and (*d*) KS with $\lambda = 0$; $B = 1$: (*e*) RDT and (*f*) KS with $\lambda = 0$; $B = 10$: (*g*) RDT and (*h*) KS with $\lambda = 0$.

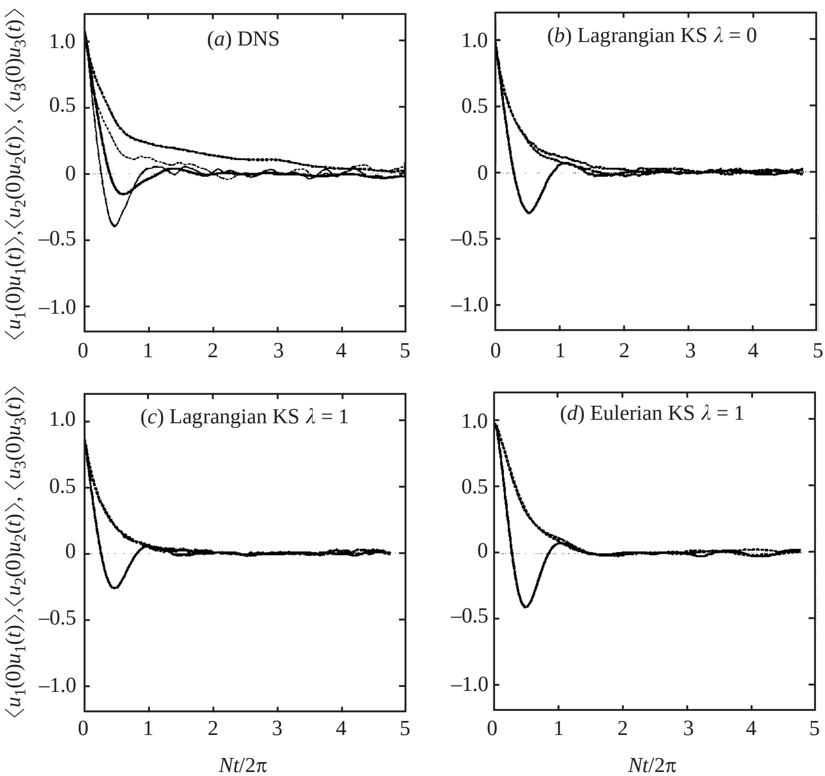

FIGURE 5. Two-time horizontal velocity correlations for the purely stratified case $B = 0$: – – – –, horizontal; ———, vertical. (*a*) DNS results for Eulerian correlations are drawn with thick lines, and thin lines for Lagrangian correlations; (*b*) Lagrangian correlations from KS with $\lambda = 0$; (*c*) Lagrangian correlations from KS with $\lambda = 1$; (*d*) Eulerian correlations from KS with $\lambda = 1$.

different values of the Rossby number $Ro = 0.04$ and 0.08, and the DNS case at $B = \infty$. To be precise, $\Omega^2\langle\tilde{x}_1^2\rangle/u'^2$ and $\Omega^2\langle\tilde{x}_3^2\rangle/u'^2$ are plotted against $\Omega t/\pi$, whereby a very good collapse of the curves is obtained. Moreover, we observe the following two features: the evolution ultimately leads to a Brownian behaviour, and the ratio of vertical to horizontal diffusivity tends towards the analytically predicted value of 2. RDT and KS with $\lambda = 1$ exhibit this value exactly, whereas in DNS this ratio only reaches the smaller value 1.4. Our DNS run exhibits the long-time Brownian behaviour as well, although for $\Omega t/u'^2 \gtrsim 10$ dispersion in the DNS seems to drop a little as observed on the zoomed figure 10(*b*), departing from the behaviour of RDT and KS. This can be due to nonlinear effects, but also to the fact that vertical and horizontal diffusivity depend on the particular realization of initial phases of the turbulent structures in the flow. There again, one is confronted by the classical problem of obtaining converged ensemble averages, which require many realizations, particularly in the case of rotating turbulence.

6.2. *General case with stable stratification*

A plateau is predicted for the r.m.s. vertical displacement, as shown by computing $\Delta_{33}(0, t)$ from (4.6) using (2.2) and the simplified Corrsin hypothesis. In so doing, $\cos\sigma t\cos\sigma t'$ and $\sin\sigma t\sin\sigma t'$ in (4.6) are replaced by $\sin^2\sigma t/\sigma^2$ and $(1-\cos\sigma t)^2/\sigma^2$, respectively. In this analytical approach, the integrals look very similar to the case of

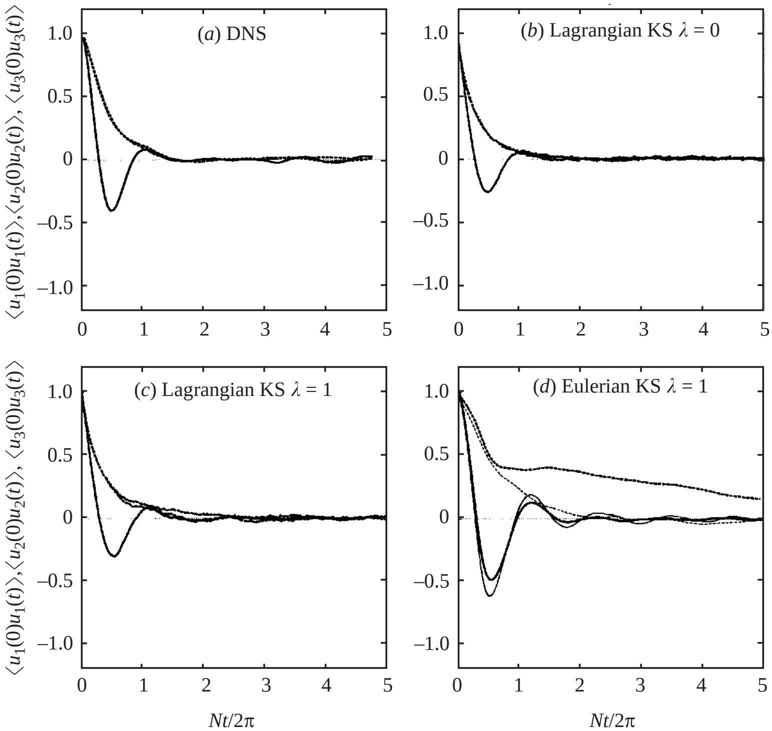

FIGURE 6. As figure 5 but for the stratification-dominant case $B = 1/10$.

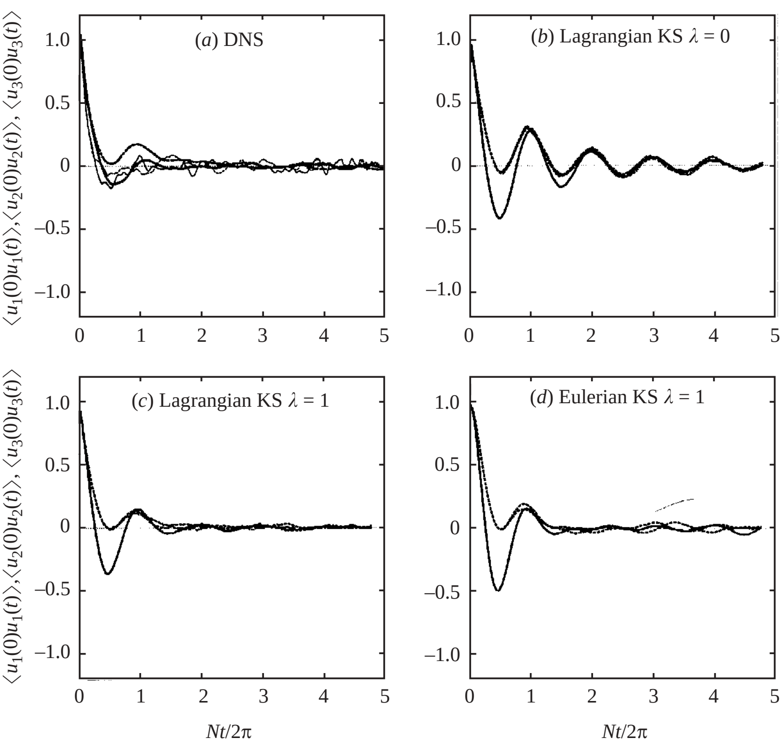

FIGURE 7. As figure 5 but for the non-dispersive case $B = 1$.

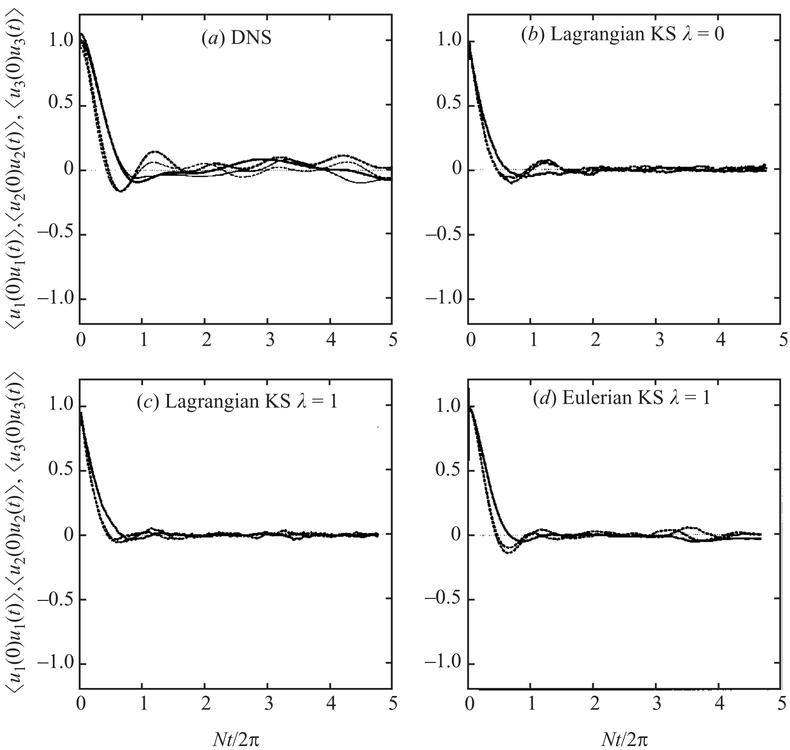

FIGURE 8. As figure 5 but for the rotation dominant case $B = 10$.

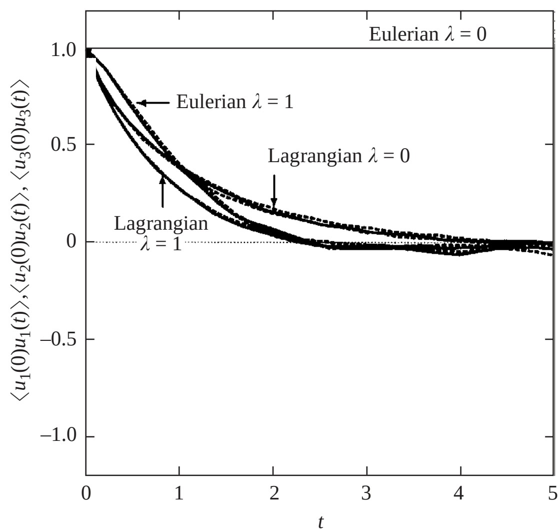

FIGURE 9. Correlations of velocity for isotropic turbulence, computed from KS: ———, vertical; – – – –, horizontal.

pure rotation, especially in the simplest case $\alpha = 1$, where $\Delta_{33}(0, t)$ at $N \neq 0$ is an integral derived from (4.9) by only replacing $2\Omega\mu$ by σ. Nevertheless, its evolution exhibits a plateau only if $N \neq 0$, illustrating the strong sensitivity of the integral to the mathematical form of the dispersion law.

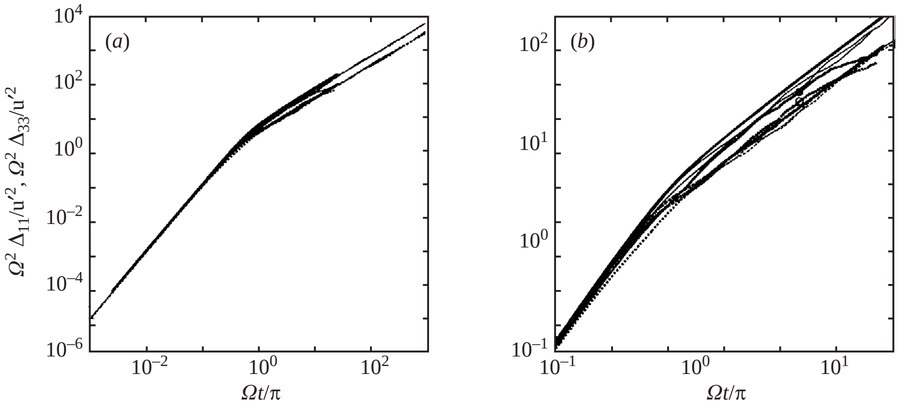

FIGURE 10. One-particle non-dimensional diffusion for pure rotation ($B = \infty$), vertical $\Omega^2\Delta_{33}/u'^2$ (——) and horizontal $\Omega^2\Delta_{11}/u'^2$ (- - - -). The thin lines (extending to $\Omega t/\pi \simeq 1000$) show results obtained with KS, with two different values of the Rossby number, $Ro = 0.08$ and 0.04; in each set, the horizontal dispersion is the lowest of the two. Heavy lines indicate analytical RDT with the simplified Corrsin hypothesis (——, vertical dispersion; - - - -, horizontal dispersion), and DNS results, identified on the close-up plot (*b*) by a black circle for the vertical dispersion, and an open circle for the horizontal dispersion. On the large-scale plot (*a*), the t^2 and t laws are easily identified, respectively at short and long times.

Looking at its horizontal counterpart, the leading term, which is obtained by replacing (4.7) in (2.2) and invoking the simplified Corrsin hypothesis, shows that the square of the horizontal displacement is proportional to $[\int_0^1 F(B,\mu)\,\mathrm{d}\mu]t^2$. Such a 'ballistic' behaviour in the horizontal direction, already found by Kaneda (2000), disagrees with KS results with $\lambda = 1$, which exhibit an eventual Brownian behaviour at largest time (Nicolleau & Vassilicos 2003), and preliminary DNS results tend to confirm this KS long-time behaviour, although many more runs would be needed for a complete confirmation.

In figure 11, we collect all the analytical results obtained from RDT with the simplified Corrsin hypothesis. An interesting feature is the presence of a transient t zone, and the larger the B the larger the zone, although the t^2 law always 'catches up', except for the pure rotation case. Horizontal dispersion in DNS contains hints of this transient t zone, in the same B range, although the time duration of this zone is much shorter.

6.3. *Physical discussion*

When the dynamics is not completely dominated by rapid oscillations connected to dispersive waves, the analytical procedure only captures a 'ballistic' regime at larger times, since quasi-steady motion prevails. This is the case for the non-rotating non-stratified flow, in which inviscid RDT yields only pure steady Eulerian correlations. It is also the case for horizontal dispersion for $N \neq 0$, since the pure divergence-free horizontal velocity field, which contributes to the quasi-geostrophic mode, disperses the particles with increasing r.m.s. displacement at increasing Nt. This effect only leads to a 'ballistic' regime when predicted by our analytical procedure, but to a more realistic 'Brownian' regime with KS (especially with $\lambda = 1$), and DNS seems to support the latter results.

Only in the case of pure rotation, $N = 0$, are all components only altered by dispersive inertial waves. Not surprisingly, the analytical procedure and KS at $\lambda = 0$

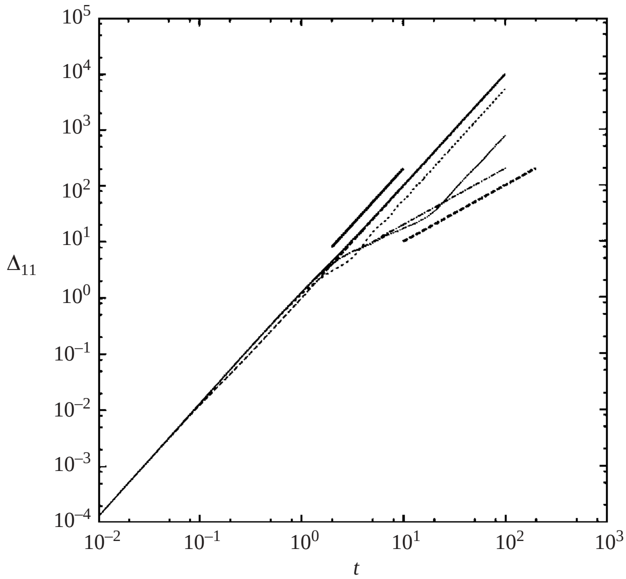

FIGURE 11. One-particle diffusion $\Delta_{11}(t) = \Delta_{22}(t)$ calculated with analytical RDT, using the simplified Corrsin hypothesis, for the different parameter cases: ——, $B = 0$; – – – –, $B = 1/10$; - - - -, $B = 1$; $\cdots$, $B = 10$; —·—·—, $B = \infty$. The two segments show t and t^2 power laws.

and $\lambda = 1$ are in agreement in this case in predicting a 'Brownian' behaviour with a constant ratio of the smaller horizontal diffusivity to the vertical diffusivity. The level of the turbulent diffusion obtained in KS depends on the Rossby and Reynolds numbers, in a way which is briefly mentioned in the next section.

Crude scaling arguments based on the Rossby radius, often used in rotating flows, yield the wrong conclusion that for horizontal dispersion a plateau may be expected (see for instance the sketch in figure 18 in Jacquin *et al.* 1990). When the nonlinearity is important, however, it could have a different impact on the vertical and on horizontal diffusivities in pure rotation, a problem which remains open and which motivates future refined DNS calculations.

Finally, it is important to underline that the theoretical and computational tools used here are completely different from the Lagrangian stochastic models. For instance, the classical approach for diffusion in stratified fluids by Csanady (1964) uses a stochastic model in which the pressure is treated as a random force, in contrast with our KS, RDT, and pseudo-spectral DNS approaches, where the pressure field is naturally accounted for by the divergenceless property of the whole flow.

7. Concluding remarks

Using analytical linearized theory, the stochastic kinematic simulation model (KS), direct numerical simulations, and comparisons thereof, we have drawn conclusions pertaining to the validity of Corrsin's hypothesis in stably stratified/rotating turbulent flows, and to the diffusion laws, summed up as follows.

We begin with our evaluation of the simplified Corrsin hypothesis by KS.

(i) We find that this hypothesis is always valid in the vertical direction, irrespective of the value of B, and for both $\lambda = 0$ and $\lambda = 1$.

(ii) In the horizontal direction, Eulerian two-time velocity correlations decay to 0 only as a result of nonlinearity (i.e. when $\lambda = 1$ which models the decorrelation due to Navier–Stokes nonlinearity) when B is not infinite. When $B = \infty$, the horizontal Eulerian two-time correlations decay to zero as a result of the dispersivity of inertial waves (and therefore for both $\lambda = 0$ and $\lambda = 1$). In the presence of stratification (i.e. at finite values of B) part of horizontal motion is not affected by inertia–gravity waves, so that the horizontal Eulerian two-time correlations do not decay to zero except when the effect of decorrelation by Navier–Stokes nonlinearity is factored into the KS model by $\lambda = 1$. However, Lagrangian two-time velocity correlations decay to 0 even when the nonlinearity of the velocity field is severely depleted, that is even when $\lambda = 0$ in KS, because of the time-decorrelation that is inherent in the integration of particle trajectories. Hence, the simplified Corrsin hypothesis is not valid in the horizontal when the nonlinearity is depleted in the flow itself. In the rotating case without stratification, the simplified Corrsin hypothesis is always valid in the horizontal as a result of the dispersive inertial waves that affect the whole flow.

Our DNS results do not invalidate the simplified Corrsin hypothesis in the vertical in all cases of stratification with or without rotation. These results are not conclusive for the case of pure rotation in the vertical direction. In the horizontal direction, all our DNS runs with dominant stratification show that the simplified Corrsin hypothesis fails after a short time. This might suggest that the impact of Navier–Stokes nonlinearity on the validity of this hypothesis is complex and not easy to model: the simple nonlinear model incorporated in KS through random frequencies, with $\lambda = 1$, seems to be sufficient to validate the simplified Corrsin hypothesis for horizontal motion, but DNS results are not in agreement for long enough time. Long-time behaviour being the hardest to compute accurately both in KS and DNS of rotating and/or stratified turbulence (and in fact more so in DNS), we chose to take a careful stance and not draw definitive conclusions from this failure of DNS to fully support the simplified Corrsin hypothesis in the horizontal direction.

It is a direct consequence of Taylor's (1921) relation that any mechanical effect producing negative loops in two-time Lagrangian correlations can reduce turbulent diffusion. In the context of this paper, these effects are caused by the stratification and/or rotation and are modulated, sometimes even prevented, by wave dispersivity. Accordingly, concerning anisotropic diffusivity, we have observed the following.

(i) In the case of rotation without stratification, RDT and the simplified Corrsin hypothesis along with Taylor's (1921) relation imply that both the horizontal and vertical diffusion is Brownian for long times and ballistic for short times and that the ratio of the vertical to the horizontal turbulent diffusivities is 2. KS confirms the ballistic and Brownian behaviours, as well as the ratio 2, but predicts a different scaling which depends on both the Rossby and Reynolds numbers. KS results suggest a scaling of $\langle \tilde{x}^2 \rangle / L^2 \sim Ro^2 u' t / L$ (Nicolleau & Vassilicos 2003), in contrast to the $Ro\, u' t / L$ dependence which derives from the analytical relationship (4.13).

(ii) In the case of stratification with or without rotation, RDT and the simplified Corrsin hypothesis along with Taylor's (1921) relation imply that the horizontal diffusion is ballistic for short and long times with a Brownian behaviour in between. This theory predicts that the vertical diffusion is depleted at long times (still ballistic at short ones) to the point there being a zero vertical turbulent diffusivity if the effects of molecular diffusivity are neglected. These results are in agreement with those of the KS with $\lambda = 0$ of Nicolleau & Vassilicos (2003). Their KS with $\lambda = 1$ leads to the same results except in the horizontal direction where they find a Brownian long-term behaviour of turbulent diffusion. This distinction between KS results for $\lambda = 1$ and

$\lambda = 0$ is in keeping with the fact that the simplified Corrsin hypothesis is validated by KS in the horizontal for $\lambda = 1$ but not for $\lambda = 0$.

J. C. V. acknowledges support from the Royal Society. Direct numerical simulations were performed on the Fujitsu VPP5000 of CEA, thanks to computing time allocated by the "Centre Grenoblois de Calcul Vectoriel."

Appendix. Analytical or semi-analytical results for damped oscillations

The Craya–Herring frame of reference is defined by

$$\boldsymbol{e}^1 = \frac{\boldsymbol{k} \times \boldsymbol{n}}{|\boldsymbol{k} \times \boldsymbol{n}|}, \quad \boldsymbol{e}^2 = \frac{\boldsymbol{k}}{k} \times \boldsymbol{e}^1, \quad \boldsymbol{e}^3 = \frac{\boldsymbol{k}}{k}, \tag{A 1}$$

except for $\boldsymbol{k} \| \boldsymbol{n}$, where it coincides with the fixed frame of reference. Note that the spectral components of the velocity/temperature vector $\hat{\boldsymbol{w}} = \hat{w}_1 \boldsymbol{e}^1 + \hat{w}_2 \boldsymbol{e}^2 + \hat{w}_3 \boldsymbol{e}^3$ written in this frame have counterparts in physical space which can be directly associated with the oscillating motion and the potential vorticity. Accordingly, w^2 represents twice the total, kinetic + potential, energy.

From (4.3), (4.4) and (4.5) one finds

$$V_{ii}(\boldsymbol{k}, t, t') = \frac{E(k, 0)}{8\pi k^2} \left[1 - (1-\alpha)\frac{\sigma_r^2}{\sigma^2} + \left(2 - (1-\alpha)\frac{\sigma_s^2}{\sigma^2} \right) \cos\sigma(t - t') \right]. \tag{A 2}$$

From (A 1) $\hat{u}_3 = \hat{w}_2 e_3^2$ and $\hat{w}_3 = \mathrm{i}N^{-1}\hat{b}$, with $n_i = \delta_{i3}$, $e_3^1 = 0$ and $e_3^2 = -\sin\theta_k$.

From (4.3) $\hat{w}_2 = \hat{w}_2^{t=0} \cos\sigma t - (\sigma_r \hat{w}_1^{t=0} + \sigma_s \hat{w}_3^{t=0}) \sin\sigma t/\sigma$, so that using (4.4) and (4.5)

$$V_{22}(\boldsymbol{k}, t, t') = \frac{E}{8\pi k^2} \left[\cos\sigma t \cos\sigma t' + \frac{\sigma_r^2 + \alpha\sigma_s^2}{\sigma^2} \sin\sigma t \sin\sigma t' \right], \tag{A 3}$$

from which the integrand of (4.6), or $(1 - \mu^2)V_{22}$, is derived. From $\hat{w}_3 = \hat{w}_1^{t=0}\sigma_r\sigma_s/\sigma^2(\cos\sigma t - 1) + \hat{w}_3^{t=0}(\sigma_r^2 + \sigma_s^2\cos\sigma t)/\sigma^2 + \hat{w}_2^{t=0}(\sigma_s/\sigma)\sin\sigma t$ one obtains

$$V_{33}(\boldsymbol{k}, t, t') = \frac{E}{8\pi k^2} \left[H(B, x) - (1-\alpha)\frac{\sigma_r^2\sigma_s^2}{\sigma^4}(\cos\sigma t + \cos\sigma t') \right.$$
$$\left. + L(B, x)\cos\sigma t \cos\sigma t' + \frac{\sigma_s^2}{\sigma^2}\sin\sigma t \sin\sigma t' \right]. \tag{A 4}$$

Finally, V_{11} is obtained by subtracting (A 3) and (A 4) from (A 2), so that

$$V_{11}(\boldsymbol{k}, t, t') = \frac{E}{8\pi k^2} \left[F(B, \mu) + (1-\alpha)\frac{\sigma_r^2\sigma_s^2}{\sigma^4}(\cos\sigma t + \cos\sigma t') \right.$$
$$\left. + C(B, \mu)\cos\sigma t \cos\sigma t' + \frac{\sigma_r^2}{\sigma^2}\sin\sigma t \sin\sigma t' \right], \tag{A 5}$$

with

$$\left.\begin{aligned} H(B,\mu) &= \sigma_r^2(\sigma_s^2 + \alpha\sigma_r^2)/\sigma^4, \quad L(B,\mu) = \sigma_s^2(\sigma_r^2 + \alpha\sigma_s^2)/\sigma^4 \\ F(B,\mu) &= \sigma_s^2(\sigma_s^2 + \alpha\sigma_r^2)/\sigma^4, \quad C(B,\mu) = \sigma_r^2(\sigma_r^2 + \alpha\sigma_s^2)/\sigma^4, \end{aligned}\right\} \tag{A 6}$$

and (4.7) is derived from its integrand $V_{11} + \mu^2 V_{22}$.

Only single-time correlations for $t' = t$ are considered from here on. In RDT results for single-time Eulerian correlations, we shall take advantage of two conservation laws: conservation of *total energy* in any orthonormal frame of reference $w_i^* w_i = \sum_{\epsilon=0,\pm1} \xi^{\epsilon *}\xi^{\epsilon}$, and conservation of *QG energy* $\xi^{0*}\xi^{0}$.

Finally $\langle u_3^2 \rangle$ and $\langle b^2 \rangle$ are given by integrals over the angular variable μ. Their history shows damped oscillations (due to the integral of $e^{mi\sigma t}, m = \pm 1, \pm 2$) and constant terms ($m = 0$ due to contributions from $\xi^{0*}\xi^{0}$, $\xi^{1*}\xi^{1}$, $\xi^{-1*}\xi^{-1}$). Hence, at least the asymptotic values can be analytically calculated. Of course, other simplifications due to semi-axisymmetry (axisymmetry without mirror symmetry) have to be used ($\langle u_1^2 \rangle = \langle u_2^2 \rangle$, etc.), as well as optimal sums and differences of solution equations. Reynolds stress components and buoyancy variance are

$$\langle u_3^2 \rangle = \frac{2E_{kin}(0)}{3}[P + (1-\alpha) f_v(Nt, B)] \quad \text{(A 7)}$$

$$N^{-2}\langle b^2 \rangle = \frac{2E_{kin}(0)}{3}[Q + (1-\alpha) f_b(Nt, B)] \quad \text{(A 8)}$$

$$\langle u_1^2 \rangle + \langle u_2^2 \rangle = \frac{2E_{kin}(0)}{3}[3(1+\alpha/2) - P - Q - (1-\alpha)(f_v(Nt, B) + f_b(Nt, B))] \quad \text{(A 9)}$$

with

$$P = \frac{3}{4}\int_0^1 \frac{1-\mu^2}{B^2\mu^2 + (1-\mu^2)}[2B^2\mu^2 + (1-\mu^2)(1+\alpha)]\,\mathrm{d}\mu, \quad \text{(A 10)}$$

$$Q = \frac{3}{4}\int_0^1 \left(1 - (1-\alpha)\frac{B^4\mu^4 + (1-\mu^2)^2/2}{(B^2\mu^2 + 1 - \mu^2)^2}\right)\mathrm{d}\mu, \quad \text{(A 11)}$$

$$f_v(Nt, B) = \frac{3}{4}\int_0^1 \frac{(1-\mu^2)^2}{B^2\mu^2 + (1-\mu^2)}\cos[2Nt\sqrt{B^2\mu^2 + (1-\mu^2)}]\,\mathrm{d}\mu, \quad \text{(A 12)}$$

$$f_b(Nt, B) = -\frac{3}{4}\int_0^1 \left(\frac{(1-\mu^2)^2}{2(B^2\mu^2 + 1 - \mu^2)^2}\cos[2Nt\sqrt{B^2\mu^2 + (1-\mu^2)}] + \frac{2B^2\mu^2(1-\mu^2)}{(B^2\mu^2 + 1 - \mu^2)^2}\cos[Nt\sqrt{B^2\mu^2 + (1-\mu^2)}]\right)\mathrm{d}\mu. \quad \text{(A 13)}$$

$P(\alpha, B)$ and $Q(\alpha, B)$ are easy to calculate, especially for $B = 0$ (no rotation), $B = 1$ (no wave dispersivity), pure rotation (singular case). For instance $P = (1+\alpha)/2$ for $B = 0$, and $P = (3 + 2\alpha)/5$ for $B = 1$. The function $f(Nt, B)$ can be shown to give damped oscillations.

REFERENCES

Babin, A., Mahalov, A. & Nicolaenko, B. 1998 On nonlinear baroclinic waves and adjustment of pancake dynamics. *Theor. Comput. Fluid Dyn.* **11**, 215–235.

Bartello, P. 1995 Geostrophic adjustment and inverse cascades in rotating stratified turbulence. *J. Atmos. Sci.* **52**, 4410–4428.

Britter, R. E., Hunt, J. C. R., Marsh, G. L. & Snyder, W. H. 1983 The effect of stable stratification on turbulent diffusion and the decay of grid turbulence. *J. Fluid Mech.* **127**, 27–44.

Cambon, C. 2001 Turbulence and vortex structures in rotating and stratified flows. *Eur. J. Mech.* B *Fluids* **20**, 489–510.

Corrsin, S. 1963 Estimates of the relations between Eulerian and Lagrangian scales in large Reynolds number turbulence. *J. Atmos. Sci.* **20**, 115–119.

Csanady, G. T. 1964 Turbulent diffusion in a stratified fluid. *J. Atmos. Sci.* **21**, 439–447.

Godeferd, F. S. & Cambon, C. 1994. Detailed investigation of energy transfers in homogeneous stratified turbulence. *Phys. Fluids* **6**, 2084–2100.

Hanazaki, H. 2002 Linear processes in stably and unstably stratified rotating turbulence. *J. Fluid Mech.* **465**, 157–190.

Hanazaki, H. & Hunt, J. C. R. 1996 Linear processes in unsteady stably stratified turbulence. *J. Fluid Mech.* **318**, 303–337.

Iida, O. & Nagano, Y. 1999 Coherent structure and heat transfer in geostrophic flow under density stratification. *Phys. Fluids* **11**, 368–377.

Jacquin, L., Leuchter, O., Cambon, C. & Mathieu, J. 1990 Homogeneous turbulence in the presence of rotation. *J. Fluid Mech.* **220**, 1–52.

Kaneda, Y. 2000 Single-particle diffusion in strongly stratified and/or rapidly rotating turbulence. *J. Phys. Soc. Japan* **69**, 3847–3852.

Kaneda, Y. & Ishida, T. 2000 Suppression of vertical diffusion in strongly stratified turbulence. *J. Fluid Mech.* **402**, 311–327.

Kimura, Y. & Herring, J. R. 1996 Diffusion in stably stratified turbulence. *J. Fluid Mech.* **328**, 253–269.

Kimura, Y. & Herring, J. R. 1999 Particle dispersion in rotating stratified turbulence. In *Proc. FEDSM99 3rd ASME/JSME Joint Fluid Engineering Conf.* FEDSM99-7753.

Kraichnan, R. H. 1970 Diffusion by a random velocity field. *Phys. Fluids* **13**, 22–31.

Kraichnan, R. H. 1977 Lagrangian velocity covariance in helical turbulence. *J. Fluid Mech.* **81**, 385–398.

Lundgren, T. S. & Pointin, Y. B. 1976 Turbulent self-diffusion. *Phys. Fluids* **19**, 355–358.

Mann, J. & Ott, S. 2002 An experimental test of Corrsin's independence hypothesis and related assumptions. Number XXVII in General Assembly, Nice, France. EGS.

Nicolleau, F. & Vassilicos, J. C. 2000 Turbulent diffusion in stably stratified non-decaying turbulence. *J. Fluid Mech.* **410**, 123–146.

Nicolleau, F. & Vassilicos, J. C. 2003 Kinematic simulation for stably stratified and rotating turbulence. *J. Fluid Mech.* (to be submitted).

Orszag, S. A. & Patterson, G. S. 1972 Numerical simulation of three-dimensional homogeneous isotropic turbulence. *Phys. Rev. Lett.* **28**, 76–79.

Riley, J. J., Metcalfe, R. W. & Weissman, M. A. 1981 Direct numerical simulations of homogeneous turbulence in density-stratified fluids. In *Proc. AIP Conf. on Nonlinear Properties of Internal Waves* (ed. B. J. West), pp. 79–112. American Institute of Physics.

Taylor, G. I. 1921 Diffusion by continuous movements. *Proc. Lond. Math. Soc.* (2) **20**, 196–211.

Vincent, A. & Meneguzzi, M. 1991 The spatial structure and statistical properties of homogeneous turbulence. *J. Fluid Mech.* **225**, 1–20.

Vincent, A., Michaud, G. & Meneguzzi, M. 1996 On the turbulent transport of a passive scalar by anisotropic turbulence. *J. Fluid Mech.* **8**, 1312–1320.

Yeung, P. K. 2002 Lagrangian investigations of turbulence. *Annu. Rev. Fluid Mech.* **34**, 115–142.